ATKINSON SCIENCE

THEORY GUIDE

Prandtl-Meyer Function Web Application

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22 September 2020

Atkinson Science welcomes your comments on this Theory Guide. Please send an email to keith.atkinson@atkinsonscience.co.uk.

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1 Introduction

You can find the Atkinson Science Prandtl-Meyer Function web application at the web address https://atkinsonscience.co.uk/WebApps/Aerospace/PrandtlMeyerFunction.aspx. There is a user guide that you can download at the same address. The Prandtl-Meyer function is used to calculate the change in Mach number or flow inclination angle when a supersonic flow undergoes an isentropic expansion or compression by turning. Referring to Figure 1, an isentropic expansion by turning occurs when a gas flows over a convex corner, so that the flow along the wall is turned *away* from the main flow. An isentropic compression by turning occurs when a gas flows over a concave corner, so that the flow along the wall is turned *into* from the main flow.

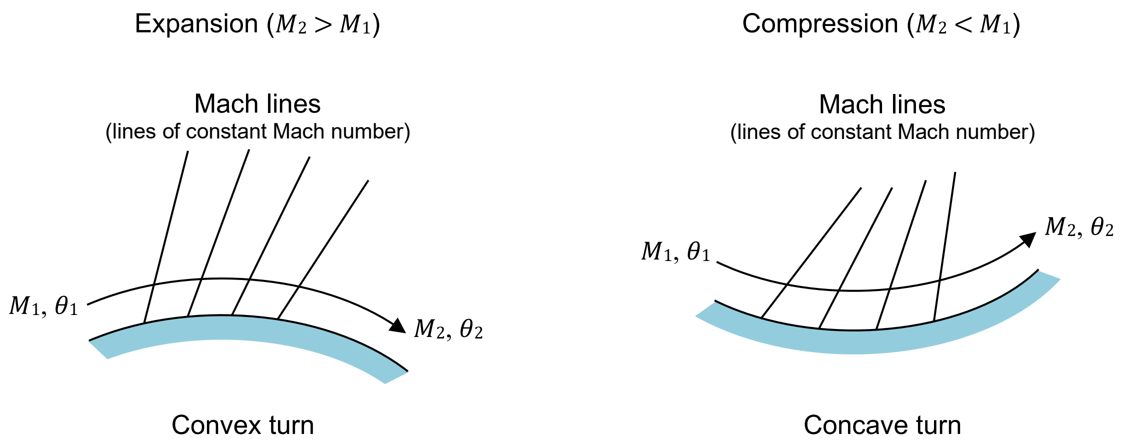
In Figure 1 the Mach number M and flow inclination angle θ change from (M_1, θ_1) to (M_2, θ_2). The Prandtl-Meyer function enables us to determine M_2 given M_1, θ_1 and θ_2, or θ_2 given M_1, θ_1 and M_2.

The function applies to a calorically perfect gas, and the derivation of the function can be found in text books on compressible flow, such as Refs. [1] and [2]. The function has the form

$$\theta = \nu(M)$$

The angle θ is chosen to be zero when $M = 1$ and increases monotonically with M. Evaluating $\nu(M)$ is laborious, and many text books, such as Refs. [1] and [2], give tables of $\nu(M)$ against M. The Prandtl-Meyer function web application is intended to replace these tables.

Figure 1 Isentropic expansion and compression by turning



1.1 Isentropic expansion

If the flow incidence angle θ_2 after an isentropic *expansion* is known, then we can calculate the Prandtl-Meyer function at exit from the corner as follows:

$$\nu_2 = \nu_1 + |\theta_2 - \theta_1|$$

where $\nu_1 = \nu(M_1)$ and $\nu_2 = \nu(M_2)$. The exit Mach number M_2 can then be determined from ν_2 using tables.

If the exit Mach number M_2 is known, then we calculate the exit incidence angle as follows:

$$|\theta_2 - \theta_1| = \nu_2 - \nu_1$$

The flow angle θ_2 can take two values, but the turning is $\nu_2 - \nu_1$ in both cases.

1.2 Isentropic compression

If the flow incidence angle θ_2 after an isentropic *compression* is known, then we can calculate the Prandtl-Meyer function at exit from the corner as follows:

$$\nu_2 = \nu_1 - |\theta_2 - \theta_1|$$

The exit Mach number M_2 can then be determined from ν_2 using tables.

If the exit Mach number M_2 is known, then we calculate the exit incidence angle as follows:

$$|\theta_2 - \theta_1| = \nu_1 - \nu_2$$

The flow angle θ_2 can take two values, but the turning is $\nu_1 - \nu_2$ in both cases.

2 Mach number from the Prandtl-Meyer function

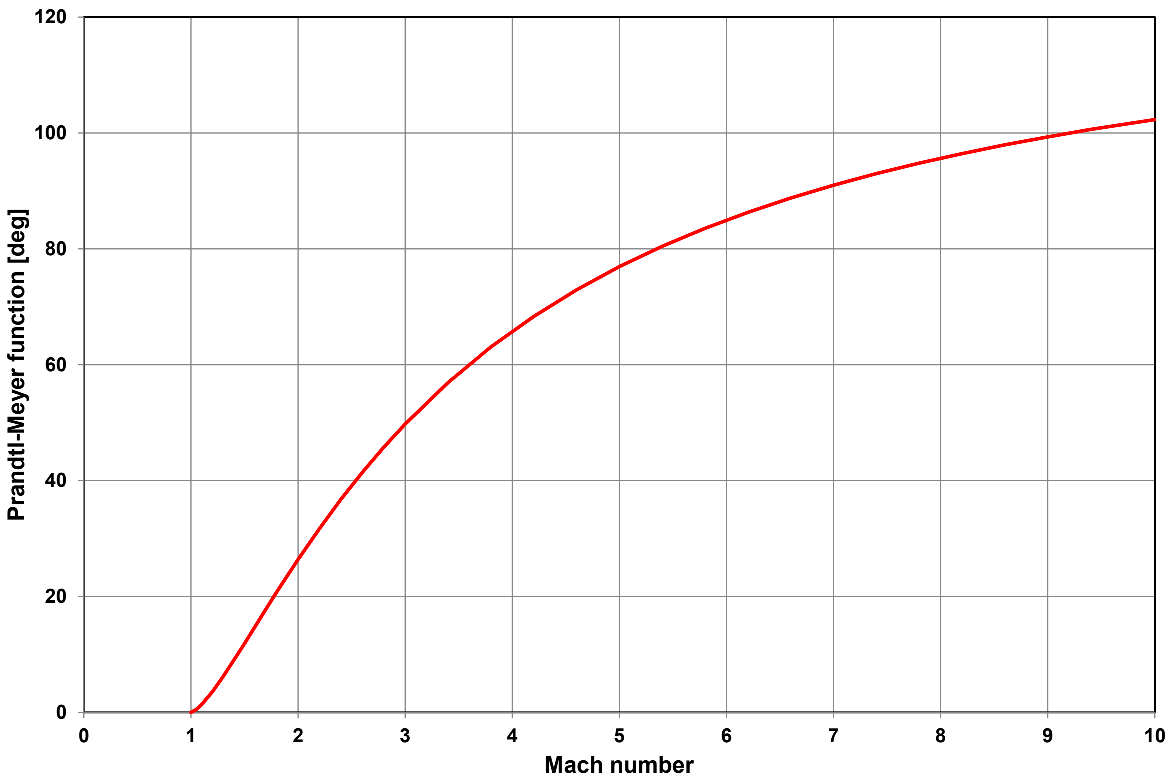
The Prandtl-Meyer function is

$$\nu(M) = \sqrt{\frac{\gamma+1}{\gamma-1}} \tan^{-1}\left(\sqrt{\frac{\gamma-1}{\gamma+1}(M^2-1)}\right) - \tan^{-1}\left(\sqrt{M^2-1}\right) \quad \text{[radians]} \quad (1)$$

where $\gamma = c_p/c_v$ is the ratio of specific heats of the gas. Tabulated values of $\nu(M)$ against M are usually for $\gamma = 1.4$, which is the ratio of specific heats of the International Standard Atmosphere (Ref. [3]).

The equation expresses ν explicitly in terms of M. The variation of ν with M is shown in Figure 2. The ratio of specific heats γ in Figure 2 is 1.4.

Figure 2 Prandtl-Meyer function



As $M \to \infty$ the arctan functions is Eqn. (1) tend to $\pi/2$ and so we can write

$$\nu(M \to \infty) = \frac{\pi}{2}\left(\sqrt{\frac{\gamma+1}{\gamma-1}} - 1\right) \quad \text{[radians]}$$

When $\gamma = 1.4$, $\nu(M \to \infty) = 130°$ (2.27 rad). This is the upper limit of ν for which it is possible to find a value of M.

To the author's knowledge there is no general solution of the inverse of the Prandtl-Meyer function for any γ. An exact solution has been published for $\gamma = 5/3$ (see Ref. [4]). In this case, the square root term involving γ is particularly simple:

$$\sqrt{\frac{\gamma + 1}{\gamma - 1}} = 2$$

However, we can calculate the inverse quite quickly and easily for the general case by using a numerical method on a computer. The Newton-Raphson iterative method is well suited to the task.

In order to use the Newton-Raphson method, we require an equation for the derivative $d\nu/dM$. We can write the Prandtl-Meyer function as follows.

$$\nu = \frac{1}{\lambda}\tan^{-1}(\lambda\beta) - \tan^{-1}(\beta)$$

where

$$\lambda = \sqrt{\frac{\gamma - 1}{\gamma + 1}}$$

and

$$\beta = \sqrt{M^2 - 1}$$

Differentiating ν with respect to β,

$$\frac{d\nu}{d\beta} = \frac{1}{\lambda}\frac{\lambda}{1 + \lambda^2\beta^2} - \frac{1}{1 + \beta^2}$$

$$= \frac{(1 - \lambda^2)\beta^2}{(1 + \beta^2)(1 + \lambda^2\beta^2)}$$

Differentiating β with respect to M,

$$\frac{d\beta}{dM} = \frac{M}{\sqrt{M^2 - 1}} = \frac{M}{\beta}$$

By the chain rule,

$$\frac{d\nu}{dM} = \frac{d\nu}{d\beta}\frac{d\beta}{dM} = \frac{(1 - \lambda^2)\beta^2}{(1 + \beta^2)(1 + \lambda^2\beta^2)} \times \frac{M}{\beta}$$

$$= \frac{(1 - \lambda^2)\beta}{M(1 + \lambda^2\beta^2)} \qquad (2)$$

since $1 + \beta^2 = M^2$.

3 Newton-Raphson method

Using Eqn. (1) we can define the function

$$f(M) = \sqrt{\frac{\gamma+1}{\gamma-1}}\tan^{-1}\left(\sqrt{\frac{\gamma-1}{\gamma+1}(M^2-1)}\right) - \tan^{-1}\left(\sqrt{M^2-1}\right) - \nu \qquad (3)$$

The Mach number M is now the root of the function $f(M)$. We can use the Newton-Raphson iterative method to find the root.

In the Newton-Raphson method we make an initial guess at the root, M_i. We then draw a tangent from the point $[M_i, f(M_i)]$. The point where this tangent crosses the M axis usually represents an improved estimate M_{i+1} of the root.

The first derivative $f'(M_i)$ at M_i is equivalent to the tangent to the point, so the new estimate M_{i+1} is given by:

$$f'(M_i) = \frac{f(M_i) - 0}{M_i - M_{i+1}}$$

Rearranging this equation gives

$$M_{i+1} = M_i - \frac{f(M_i)}{f'(M_i)}$$

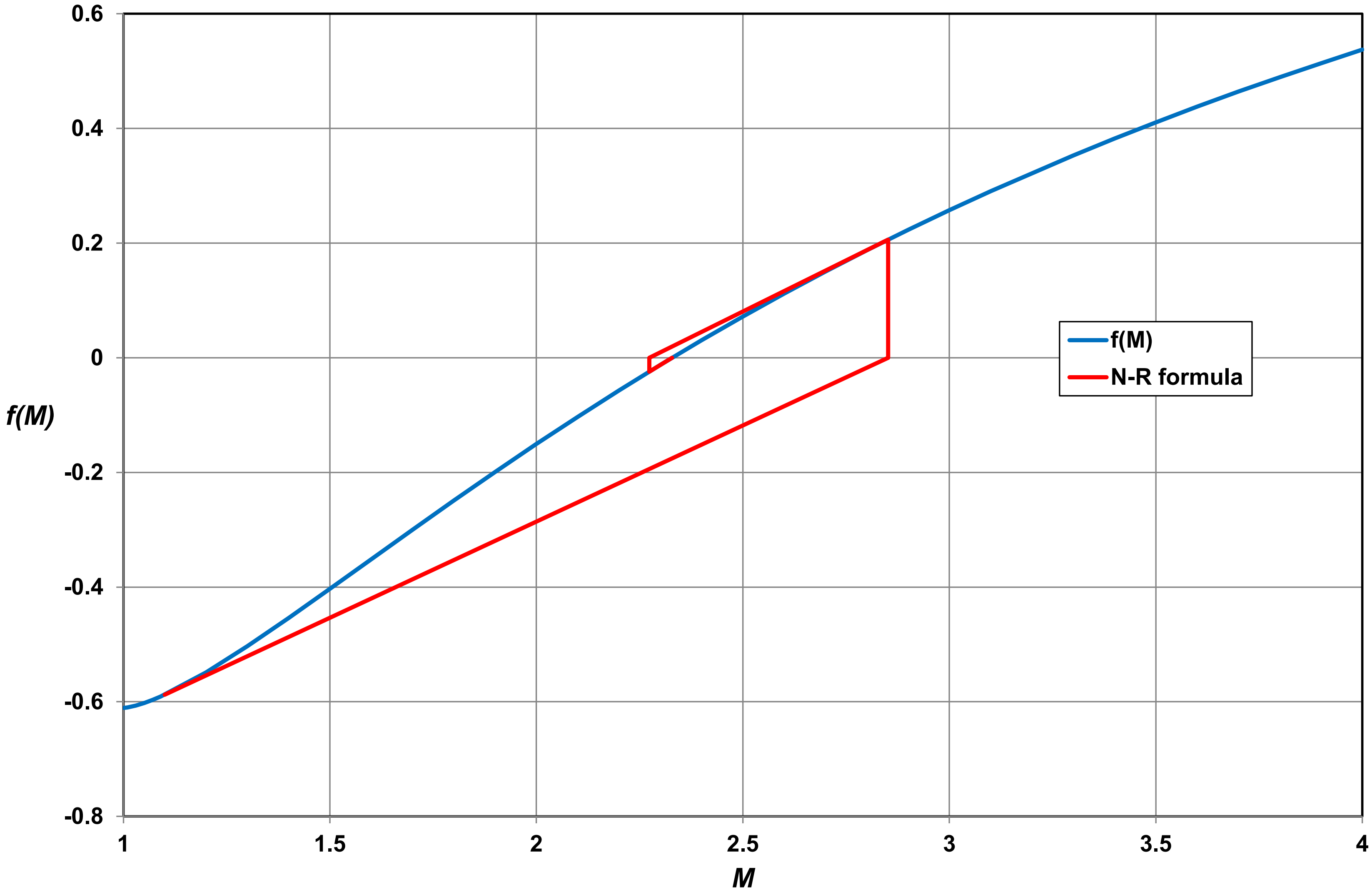
which is called the *Newton-Raphson formula*.

The last term in Eqn. (3) for $f(M)$ is a constant, so the first derivative, $f'(M)$, is given by Eqn. (2),

$$f'(M_i) = \frac{(1-\lambda^2)\beta}{M(1+\lambda^2\beta^2)}$$

Figure 5 shows a graphical depiction of the Newton-Raphson method for $\gamma = 1.4$, $\nu = 35°$ (0.6109 radians), for which $M = 2.329$. The initial guess is $M = 1.1$. We can see that the required Mach number is obtained within three iterations. In the web application we have fixed the initial guess at $M = 1.1$.

Figure 3 Graphical depiction of the Newton-Raphson method



4 Ranges of parameters

In the web application γ is 1.4, which is the ratio of specific heats for the International Standard Atmosphere (Ref. [3]). When calculating the Prandtl-Meyer function from the Mach number, the Mach number may not be less than 1. When calculating the Mach number from the Prandtl-Meyer function, the Prandtl-Meyer function may not be less than 0° or greater than 130° (2.27 radians).

5 References

1. J. D. Anderson, *Modern Compressible Flow with Historical Perspective*, 3rd Ed., McGraw-Hill, 2004.
2. H. W. Liepmann and A. Roshko, *Elements of Gasdynamics*, Dover, 2001.
3. *International Standard Atmosphere*, ISO 2533:1975, International Standards Organisation, 1975.
4. O. Ozcan, F. O. Edis, A. R. Aslan and I. Pinar, Inverse solutions of the Prandtl-Meyer function, Journal of Aircraft, Vol. 31, No. 6, 1994, pp. 1422 – 1424.